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The existence of pronormal π-Hall subgroups in E_π-groups

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To Yu.L.Ershov on his seventy-fifth birthday

Abstract

A subgroup H of a group G is called *pronormal*, if the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$. It is proven that if a finite group G possesses a π-Hall subgroup for a set of primes π, then its every normal subgroup (in particular, G itself) has a π-Hall subgroup that is pronormal in G.

Introduction

Throughout the paper the term "group" implies "finite group".

According to the definition by P. Hall, a subgroup H of a group G is called *pronormal*, if the subgroups H and H^g are conjugate in $\langle H, H^g\rangle$ for every $g \in G$.

The pronormality for subgroups is a more general property, than normality, and it plays an important role in the theory of groups. In particular, the Frattini argument holds for pronormal subgroups: *If a pronormal subgroup H of G lies in a normal subgroup A, then* $G = AN_G(H)$. This statement is often important in inductive arguments. Notice that $G = AN_G(H)$ if and only if $H^G = \{H^g \mid g \in G\}$ and $H^A = \{H^a \mid a \in A\}$ coincide.

In view of the Sylow theorem, the Sylow subgroups of a group, sa well as the Sylow subgroups of every normal subgroup are examples of pronormal subgroups. The goal of the paper is to study in what form this properties of Sylow subgroups can be reformulated for Hall subgroups. We recall the appropriate definitions.

Throughout the paper we suppose that π is a fixed set of primes. We denote by π' the set of all primes not in π; and by $\pi(n)$, the set of all prime divisors of a natural number n, while for a group G we denote the set $\pi(|G|)$ by $\pi(G)$. A natural n with $\pi(n) \subseteq \pi$ is called a π-*number*, while a group G with $\pi(G) \subseteq \pi$ is called a π-*group*. A subgroup H of G is called a π-*Hall* subgroup, if $\pi(H) \subseteq \pi$ and $\pi(|G : H|) \subseteq \pi'$. Thus in case $\pi = \{p\}$ the definition of π-Hall subgroup coincides with the usual notion of Sylow p-subgroup. A subgroup is said to be a *Hall* subgroup, if it is a π-Hall subgroup for a set of primes π, i. e. if its index and order are coprime.

According to [1] we say that G *satisfies* E_π (or briefly $G \in E_\pi$), if G possesses a π-Hall subgroup. If, at that, every two π-Hall subgroups are conjugate, then we say that G *satisfies* C_π ($G \in C_\pi$). The group satisfying E_π or C_π we call also an E_π- or a C_π-*group* respectively.

The Hall theorem implies that Hall subgroups are pronormal in solvable groups. Also the π-Hall subgroups are known to be pronormal

- in finite simple groups [2];
- in C_π-groups [3].

In [4] some properties were specified of the groups such that π-Hall subgroups exist and all are pronormal.

At the same time it is known [3] that if a set π of primes is such that $E_\pi \neq C_\pi$, then for every $X \in E_\pi \setminus C_\pi$ and $p \in \pi'$ the group $X \wr \mathbb{Z}_p$ possesses nonpronormal π-Hall subgroups, so we cannot transfer the property of pronormality of Sylow subgroups to Hall subgroups.

The following analog of the Frattini argument for π-Hall subgroups is proven in [5, Theorem 1]: *If $G \in E_\pi$, then each normal subgroup A of G possesses a π-Hall subgroup H such that $G = AN_G(H)$.*

In the paper we prove the following statement, on using results from [2, 5–9].

Theorem 1. *Let $G \in E_\pi$ for a set of primes π and $A \trianglelefteq G$. Then there is a π-Hall subgroup of A pronormal in G.*

Corollary 2. *Let π be a set of primes. Then every group possessing a π-Hall subgroup has a pronormal π-Hall subgroup.*

Notice that the result of [9] on the pronormality of π-Hall subgroups in C_π-groups (or, equivalently, on the inheriting of the C_π-property by overgroups of π-Hall subgroups) is a particular case of this statement.

Theorem 1 generalizes the above mentioned result of [5]. Notice also the following statement that generalizes the useful Lemma 6 (located below) and giving a criterion of existence of π-Hall subgroups in nonsimple groups.

Corollary 3. *Let $A \trianglelefteq G$ and let π be a set of primes. Then $G \in E_\pi$ if and only if $G/A \in E_\pi$ and A has a π-Hall subgroup H such that $H^A = H^G$.*

1 Preliminary results

The notation of the paper is standard. As we say in Introduction, π always stands for a set of primes. Given G, the set of all π-Hall subgroups of G is denoted by $\mathrm{Hall}_\pi(G)$. The notation H prn G means that H is a pronormal subgroup of G.

Lemma 4. [10, Ch. IV, (5.11)] *Let A be a normal subgroup of G. If H is a π-Hall subgroup of G, then $H \cap A$ is a π-Hall subgroup of A, while HA/A is a π-Hall subgroup of G/A.*

Recall that a group is called *π-separable*, if it has a normal series with factors either π- or π'-groups.

Lemma 5. [10, Ch. V, Theorem 3.7] *Every π-separable group satisfies C_π.*

Lemma 6. [6, Lemma 2.1(e)] *Let A be a normal subgroup of G such that G/A is a π-group, U a π-Hall subgroup of A. Then a π-Hall subgroup H of G with $H \cap A = U$ exists if and only if $U^G = U^A$.*

Lemma 7. [2, Theorem 1] *The Hall subgroups in simple groups are pronormal.*

Lemma 8. [5, Theorem 1] *Let $G \in E_\pi$ and $A \trianglelefteq G$. Then there exists $H \in \mathrm{Hall}_\pi(A)$ such that $G = AN_G(H)$. Moreover $N_G(H) \in E_\pi$ and $\mathrm{Hall}_\pi(N_G(H)) \subseteq \mathrm{Hall}_\pi(G)$.*

Lemma 9. *Let H be a subgroup of G. Considering $g \in G$, $y \in \langle H, H^g\rangle$, assume that subgroups H^y and H^g are conjugate in $\langle H^y, H^g\rangle$. Then H and H^g are conjugate in $\langle H, H^g\rangle$.*

Proof. Let $z \in \langle H^y, H^g\rangle$ and $H^{yz} = H^g$. Then $z \in \langle H, H^g\rangle$, since $\langle H^y, H^g\rangle \leqslant \langle H, H^g\rangle$. So $x = yz \in \langle H, H^g\rangle$ and $H^x = H^g$. □

The statements of the following two lemmas are evident.

Lemma 10. *Let $^{-} : G \to G_1$ be a group homomorphism and $H \leq G$. Then H prn G implies $\overline{H}$ prn $\overline{G}$.*

Lemma 11. *Let G be a group. Then H prn G implies H prn K, for every subgroup K of G such that $H \leq K$.*

Lemma 12. [2, Lemma 7] *Let G be a finite group and let $G_1, \dots, G_n$ be normal subgroups of G such that $[G_i, G_j] = 1$ for $i \neq j$ and $G = G_1 \cdot \ldots \cdot G_n$. Assume that for every $i = 1, \dots, n$ a pronormal subgroup H_i of G_i is chosen, and let $H = \langle H_1, \dots, H_n\rangle$. Then H prn G.*

Lemma 13. [7, Corollary 9] *Let $G \in E_\pi$ and $A \trianglelefteq G$. Then for every $K/A \in \mathrm{Hall}_\pi(G/A)$ there exists $H \in \mathrm{Hall}_\pi(G)$ such that $K = HA$.*

Lemma 14. *Let $H \leq G$ and $A \trianglelefteq G$. The following are equivalent:*

(1) *H prn G.*

(2) *HA prn G and H prn $N_G(HA)$.*

Proof. Assume (1). Then HA prn G by Lemma 10 and H prn $N_G(HA)$ by Lemma 11. Conversely, assume (2). Take $g \in G$. We need to show that there exists $x \in \langle H, H^g\rangle$ such that $H^x = H^g$. Since HA/A prn G/A, there exists $y \in \langle H, H^g\rangle$ with $H^yA = H^gA$. In accordance with Lemma 9 it is possible to replace H by H^y, and to assume that $HA = H^yA = H^gA$, i. e. $g \in N_G(HA)$. Since H prn $N_G(HA)$, the existence of desired x is evident. □

Lemma 15. *Let A be a π-separable normal subgroup of G and $H \in \mathrm{Hall}_\pi(A)$. Then H prn G.*

Proof. The lemma follows since the subgroup $\langle H, H^g\rangle \leq A$ is π-separable for every $g \in G$, while if H and H^g are its π-Hall subgroups, then they are conjugate in $\langle H, H^g\rangle$ by Lemma 5. □

Lemma 16. *Let B be a normal subgroup of a finite group G. Then for every normal subgroup A of G including B, and for every $H \in \mathrm{Hall}_\pi(A)$ the conditions*

(1) *HB/B prn G/B;*

(2) *$(H \cap B)$ prn B;*

(3) *$(H \cap B)^G = (H \cap B)^B$*

imply H prn G..

Proof. Since HB/B prn G/B, we have HB prn G. By Lemma 14 it is enough to show that H prn $N_G(HB)$. So, without loss of generality, we may assue that $G = N_G(HB)$, i. e. $HB \trianglelefteq G$, and $A = N_A(HB)$. Notice that in this case A/B is π-separable, since the factors A/HB and HB/B of the normal series $A \trianglerighteq HB \trianglerighteq B$ are π'- and π-groups respectively.

Take $g \in G$ arbitrary . Since $(H \cap B)^G = (H \cap B)^B$, there exists $b \in B$ such that $H^g \cap B = H^b \cap B$. Since $(H \cap B)$ prn B, there exists

$$y \in \langle H \cap B, H^b \cap B\rangle = \langle H \cap B, H^g \cap B\rangle \leq \langle H, H^g\rangle$$

such that $H^y \cap B = H^b \cap B$. By Lemma 9 the conjugacy of H and H^b in $\langle H^, H^b\rangle$ follows from the conjugacy of H^y and H^b in $\langle H^y, H^b\rangle$. Thus we can replace H by H^y and suppose that

$$(H \cap B)^g = (H \cap B)^b = (H \cap B)^y = H \cap B,$$

i. e. $g \in N_G(H \cap B)$. It is clear also that $H \leq N_A(H \cap B)$.

Since $(H \cap B)^G = (H \cap B)^B$, applying the Frattini argument we obtain $G = BN_G(H \cap B)$ and $A = BN_A(H \cap B)$. Therefore,

$$N_A(H \cap B)/N_B(H \cap B) \cong BN_A(H \cap B)/B = A/B$$

is π-separable. The group $N_B(H \cap B)$ is also π-separable, since it has the normal π-Hall subgroup $H \cap B$. Hence, $N_A(H \cap B)$ is π-separable as well. Then from $H \in \mathrm{Hall}_\pi(N_A(H \cap B))$ by Lemma 15 applied to $N_G(H \cap B)$ and its normal π-separable subgroup $N_A(H \cap B)$ we have H prn $N_G(H \cap B)$. Since $g \in N_G(H \cap B)$, for some $x \in \langle H, H^g\rangle$ the equality $H^x = H^g$ holds. Thus, H prn G. □

2 Proof of the main results

Proof of Theorem 1. Let $G \in E_\pi$ and $A \trianglelefteq G$. We need to show that A has a π-Hall subgroup H such that H prn G. We proceed by induction on $|G|$.

If $|G| = 1$, we have nothing to prove.

Let $|G| > 1$. Choose a minimal normal subgroup B of G lying in A (note that the inequality $B \neq A$ is not assumed here). Since by Lemma 4 $G/B \in E_\pi$, the factor group A/B has a π-Hall subgroup K/B such that K/B prn G/B by induction. By Lemma 8 it follows that B possesses a π-Hall subgroup V such that $G = BN_G(V)$ or, equivalently, $V^G = V^B$. This means, in particular, that $V^K = V^B$ and, by Lemma 6, there exists $H \in \mathrm{Hall}_\pi(K)$ such that $V = H \cap B$. Notice that $|A : H| = |A : K||K : H|$ is a π'-number, and so $H \in \mathrm{Hall}_\pi(A)$. Let us show that H prn G, so proving the theorem. We use Lemma 16. By the choice of K we have $HB/B = K/B$ prn G/B, which is equivalent to $HB = K$ prn G, and so (1) of Lemma 16 holds. Now B is a direct product of simple groups $B = S_1 \times \cdots \times S_n$, since B is a minimal normal subgroup of G, and $V = \langle V \cap S_i \mid i = 1, \ldots, n\rangle$.

Since by Lemma 4, $V \cap S_i \in \mathrm{Hall}_\pi(S_i)$ for every $i = 1, \ldots, n$, and by Lemma 7, $(V \cap S_i)$ prn S_i, applying Lemma 12 we obtain $H \cap B = V$ prn B, and so (2) of Lemma 16 holds. Finally, $H \cap B = V$ and by of the choice of V in B we have $(H \cap B)^G = (H \cap B)^B$. So (3) of Lemma 16 holds. Thus H prn G by Lemma 16. □

Proof of Corollary 2. The corollary is immediate from Theorem 1 for the case $G = A$. □

Proof of Corollary 3. Let $A \trianglelefteq G$. Assume that $G/A \in E_\pi$ and A has a π-Hall subgroup H such that $H^A = H^G$. Show that $G \in E_\pi$. Choose $X/A \in \mathrm{Hall}_\pi(G/A)$. Since $A \leq X \leq G$, we have
$$H^A \subseteq H^X \subseteq H^G;$$
whence $H^A \subseteq H^X$. Taking into account Lemma 6 and the fact that X/A is a π-group, we obtain $X \in E_\pi$. Since $|G : X| = |G/A : X/A|$ is a π'-number, we have
$$\varnothing \neq \mathrm{Hall}_\pi(X) \subseteq \mathrm{Hall}_\pi(G) \quad \text{and} \quad G \in E_\pi.$$

Conersely, let $G \in E_\pi$. Then by Lemma 4 $G/A \in E_\pi$. By Theorem 1, there exists a subgroup $H \in \mathrm{Hall}_\pi(A)$ such that H prn G. In particular, for every $g \in G$ there exists $a \in \langle H, H^g \rangle \leq A$ such that $H^g = H^a$. So $H^A = H^G$. $\square$

3 Remarks

Remark 1. Theorem 1 guarantees the existence of a pronormal π-Hall subgroup in every normal subgroup A of $G \in E_\pi$. The condition $G \in E_\pi$ cannot be replaced by $A \in E_\pi$ which is weaker in view of Lemma 4. Indeed, let $\pi = \{2, 3\}$ and $A = \mathrm{GL}_3(2) = \mathrm{SL}_3(2)$. Then (cм. [11, Theorem 1.2]) A has exactly the two classes of conjugate π-Hall subgroups with representatives
$$H_1 = \begin{pmatrix} \boxed{\mathrm{GL}_2(2)} & * \\ 0 & \boxed{1} \end{pmatrix} \quad \text{and} \quad H_2 = \begin{pmatrix} \boxed{1} & * \\ 0 & \boxed{\mathrm{GL}_2(2)} \end{pmatrix}$$
respectively. The class H_1^A consists of line stabilizers in the natural presentation of G, while H_2^A consists of plane stabilizers. The map $\iota : x \in A \mapsto (x^t)^{-1}$ is an automorphism of order 2 of A (here x^t is the transpose of x). This automorphism interchanges H_1^A and H_2^A. Consider the natural split extension $G = A : \langle \iota \rangle$. Subgroups H_1 and H_2 are conjugate in G. At the same time H_1 and H_2 are not conjugate in A, cotaining both H_1 and H_2, and so they are not pronormal in G. We remain to notice that $G \notin E_\pi$ in this example by Lemma 6.

Remark 2. In [3, 9] the definition of strongly pronormal subgroup is introduced. Recall that a subgroup H of G is called *strongly pronormal*, if for every $g \in G$ and $K \leq H$ there exists $x \in \langle H, K^g \rangle$ such that $K^{gx} \leq H$. In [3] the conjecture that every pronormal Hall subgroup should be strongly pronormal is formulated. In the light of Theorem 1 and its corolary it is natural to formulate the weaker conjecture: Does every E_π-group have a strongly pronormal π-Hall subgroup?

Remark 3. Lemma 16 plays an important role in the proof of Theorem 1, and from the lemma, the following test for pronormality of Hall subgroups follows in particular: Let $A \trianglelefteq G$ for a group G. A Hall subgroup H of G is pronormal if

(1) HA/A prn G/A;

(2) $(H \cap A)$ prn A;

(3) $(H \cap A)^G = (H \cap A)^A$.

The authors do not know, whether the converse is true; more precisely, whether the condition H prn G, where H is a Hall subgroup of G, implies (2) and (3)? Statement (1) follows from the pronormality of H by Lemma 10.

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